Mathematics-II (Differential Equations)
Lecture Notes
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by

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Introduction

Definition (Linear Homogeneous Partial Differential Equation of Order n)

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} +, \ldots, a_n \frac{\partial^n z}{\partial y^n} = \phi(x, y), \quad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants and $\phi(x, y)$ is any function of x and y, is called a "**homogeneous linear partial differential equation of order** n" with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Notations: We use the following notations:

$$\frac{\partial}{\partial x} = D \text{ and } \frac{\partial}{\partial y} = D'$$

Then equation (1) can be written as

$$a_0 D^n z + a_1 D^{n-1} D' z + a_2 D^{n-2} D'^2 z +, \ldots, + a_n D'^n z = \phi(x, y)$$

or

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 +, \ldots, + a_n D'^n) z = \phi(x, y)$$

or

$$F(D, D') z = \phi(x, y),$$

where
$F(D, D') = (a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 +, \ldots, + a_n D'^n)$.

Working Rule to find Complementary Functions:

Step-I: Put the given equation in the standard form

$$(a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}D'^2 +, \ldots, +a_nD'^n)z = \phi(x, y) \quad (2)$$

Step-II: Replacing D by m and D' by 1 in the equation (2), we obtain auxiliary equation (A.E.) as

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} +, \ldots, +a_n = 0 \quad (3)$$

Step-III: Solve equation (3) for m.Then following cases will be arises:

Case-1: Let $m = m_1, m_2, ..., m_n$ are different roots, then complementary function (C.F.) will be

$$C.F. = f_1(y + m_1x) + f_2(y + m_2x) + ... + f_n(y + m_nx),$$

where $f_1, f_2, ..., f_n$ are arbitrary functions.

Case-2: Let r roots $m = m_1 = m_2 = ... = m_r, (r \leq n)$ are equal, then complementary function (C.F.) will be

$$C.F. = f_1(y+mx)+xf_2(y+mx)+x^2f_3(y+mx)+...+x^{r-1}f_r(y+mx).$$

Case-3: Corresponding to a non-repeated factor D, the C.F. is taken as $f_1(y)$.

Case-4: Corresponding to a repeated factor D^r, the C.F. is taken as

$$f_1(y) + xf_2(y) + x^2f_3(y) + ... + x^{r-1}f_r(y).$$

Case-5: Corresponding to a non-repeated factor D', the C.F. is taken as $f_1(x)$.
Case-6: Corresponding to a repeated factor D'^r, the C.F. is taken as

$$f_1(x) + yf_2(x) + y^2 f_3(x) + ... + y^{r-1} f_r(x).$$

Notations: We use the following notations

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}.$$

Example

Solve $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x \partial y^2} + 6\dfrac{\partial^3 z}{\partial y^3} = 0$

Solution: The given partial differential equation can be written as

$$(D^3 - 7DD'^2 + 6D'^3)z = 0.$$

By replacing D by m and D' by 1, the auxiliary equation is

$$m^3 - 7m + 6 = 0 \implies (m-1)(m-2)(m+3) = 0.$$

Hence the roots are $m = 1, 2, -3$, which are different. Therefore general solution will be

$$z = f_1(y + x) + f_2(y + 2x) + f_3(y - 3x),$$

where f_1, f_2, f_3 are arbitrary functions.

Example

Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

Solution: By replacing D by m and D' by 1, the auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0 \implies (m-1)(m-2)(m-3) = 0.$$

Hence the roots are $m = 1, 2, 3$, which are different. Therefore general solution will be

$$z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x),$$

where f_1, f_2, f_3 are arbitrary functions.

Example

Solve the partial differential equation $25r - 40s + 16t = 0$

Solution: Given equation can be written as

$$(25D^2 - 40DD' + 16D'^2)z = 0.$$

By replacing D by m and D' by 1, the auxiliary equation is

$$25m^2 - 40m + 16 = 0 \implies (5m-4)^2 = 0.$$

Hence the roots are $m = 4/5, 4/5$, which are repeated.
Therefore general solution will be

$$z = f_1(y + \frac{4}{5}x) + xf_2(y + \frac{4}{5}x)$$

or

$$z = f_1(5y + 4x) + xf_2(5y + 4x)$$

where f_1, f_2, f_3 are arbitrary functions.

Example

Solve the partial differential equation $D^2D'^2(D+D')z=0$

Solution: The solution corresponding to the factor D^2 is $f_1(y)+xf_2(y)$
The solution corresponding to the factor D'^2 is $f_3(x)+yf_4(x)$
The solution corresponding to the factor $(D+D')$ is $f_5(y-x)$
Hence the general solution will be

$$z=f_1(y)+xf_2(y)+f_3(x)+yf_4(x)+f_5(y-x).$$

Exercise

Solve the following PDE:

(1) $(4D^2 + 12DD' + 9D'^2)z = 0$

(2) $(D^3 - 4D^2D' + 4DD'^2)z = 0$

(3) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$

(4) $r = a^2t$

(5) $2r + 5s + 2t = 0$

Thanks !!!