Mathematics-II (Differential Equations)
Lecture Notes
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by

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General solution of one-dimensional wave (vibrational) equation satisfying the given boundary conditions

Consider one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with boundary conditions $u(0,t) = 0$ and $u(a,t) = 0, \quad \forall t$.

Solution: Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

with boundary conditions $u(0, t) = 0$ and $u(a, t) = 0$.
Let the given equation has the solution of the form $u(x, t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$.

Putting these values in given equation, we have

$$X''T = \frac{1}{c^2} X T'' \implies \frac{X''}{X} = \frac{T''}{c^2 T}, \tag{2}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T''}{c^2 T} = k(constant) \implies X'' - kX = 0 \text{ and } T'' - c^2 kT = 0$$

These are ordinary differential equation of second order with constant coefficient. Now to solve these two equations $X'' - kX = 0$ and $T'' - c^2 kT = 0$, three cases arises:

Case-I When $k = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

and

$$T'' = 0 \implies T = a_3 t + a_4.$$

Thus the required solution is

$$u(x,t) = (a_1 x + a_2)(a_3 t + a_4). \tag{3}$$

Case-II When $k > 0$, we can take $k = \lambda^2 (say)$, then both equations reduces to

$$X'' - \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 - \lambda^2) = 0 \implies m = \pm\lambda. \text{ Therefore its solution will be}$$
$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$$

and

$$T'' - c^2\lambda^2 T = 0 \implies T = b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}.$$

Thus the required solution is

$$u(x,t) = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}). \tag{4}$$

Case-III When $k < 0$, we can take $k = -\lambda^2 (say)$, then both equations reduces to

$$X'' + \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i. \text{ Therefore its solution will be}$$
$$X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

and

$$T'' + c^2\lambda^2 T = 0 \implies T = c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t).$$

Thus the required solution is

$$u(x,t) = (c_1 \cos(\lambda x) + c_2 \sin(\lambda x))(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)). \tag{5}$$

Thus the equation (3), (4) and (5) are various possible solution of the given wave equation.

Given boundary conditions are $u(0,t) = u(a,t) = 0 \quad \forall t$ In view of the boundary condition, the solution given by the equation (3) becomes

$$0 = a_2(a_3t + a_4) \quad \text{and} \quad 0 = (a_1a + a_2)(a_3t + a_2)$$

$$\implies a_2 = 0 \quad \text{and} \quad (a_1a + a_2) = 0 \implies a_1 = a_2 = 0$$

Hence $u(x,t) = 0 \quad \forall t.$ This is a trivial solution.

Again, in view of the boundary condition, the solution given by the equation (4) becomes

$$0 = (b_1 + b_2)(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}) \quad \text{and}$$
$$0 = (b_1 e^{\lambda a} + b_2 e^{-\lambda a})(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t})$$

$$\implies (b_1 + b_2) = 0 \quad \text{and} \quad b_1 e^{\lambda a} + b_2 e^{-\lambda a} = 0$$
$$\implies b_1 = b_2 = 0$$

Hence $u(x, t) = 0 \quad \forall t$. This is also a trivial solution.

Finally, in view of the boundary condition, the solution given by the equation (5) becomes

$$0 = c_1(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)) \quad \text{and}$$
$$0 = (c_1 \cos(\lambda a) + c_2 \sin(\lambda a))(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t))$$

$$\implies c_1 = 0 \quad \text{and} \quad c_2 \sin \lambda a = 0$$

Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$\implies \sin \lambda a = 0 \implies \lambda a = n\pi \quad n = 1, 2, 3, ...$$

Thus $\lambda = \dfrac{n\pi}{a}, n = 1, 2, 3, ...$

Hence the solution given by the equation (5) becomes

$$u_n(x,t) = c_2 \sin\frac{n\pi}{a}\left(c_3 \cos\frac{n\pi ct}{a} + c_4 \sin\frac{n\pi ct}{a}\right)$$
$$n = 1, 2, 3, ...$$

$$u_n(x,t) = \sin\frac{n\pi}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right)$$
$$n = 1, 2, 3, ...$$

Where $E_n = (c_2c_3)$ and $F_n = (c_2c_4)$ are new arbitrary constants.

Since the given wave equation is linear, its most general solution is obtained by applying the principle of superposition, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right) \quad n = 1, 2, 3, ...$$

General solution of one-dimensional wave (vibrational) equation satisfying the given boundary and initial conditions

Consider one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where $u(x,t)$ is the deflection of the string. the solution of this equation shows how the string moves. More precisely, if the ends of string are fixed at $x = 0$ and $x = a$, we have the two boundary conditions.

$$u(0,t) = 0 \text{ and } u(a,t) = 0, \quad \forall t.$$

The form of the motion of the string will depend on the initial deflection (deflection at $t = 0$) and on the intial velocity (velocity at $t = 0$). Denoting the initial deflection by $f(x)$ and initial velocity by $g(x)$, we get two initial conditions

$$u(x,0) = f(x), \qquad 0 \leq x \leq a$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x), \quad i.e. \quad u_t(x,0) = g(x) \qquad 0 \leq x \leq a.$$

Solution: Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{6}$$

with boundary conditions $u(0,t) = 0$,
$u(a,t) = 0, u(x,0) = f(x)$ and $u_t(x,0) = g(x), \quad 0 \leq x \leq a$.
Let the given equation has the solution of the form
$u(x,t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and
$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$. Putting these values in given equation, we have

$$X''T = \frac{1}{c^2}XT'' \implies \frac{X''}{X} = \frac{T''}{c^2 T}, \tag{7}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T''}{c^2T} = k(constant) \implies X'' - kX = 0 \text{ and } T'' - c^2kT = 0$$

These are ordinary differential equation of second order with constant coefficient. Now to solve these two equations $X'' - kX = 0$ and $T'' - c^2kT = 0$, three cases arises:

Case-I When $k = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = a_1 x + a_2$ becomes $0 = a_1.0 + a_2$ and $0 = a_1.a + a_2 \implies a_1 = 0 = a_2$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So we reject case-I, when $k = 0$.

Case-II When $k > 0$, we can take $k = \lambda^2 (say)$, then first equations reduces to

$$X'' - \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 - \lambda^2) = 0 \implies m = \pm\lambda. \text{ Therefore its solution will be}$$
$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ becomes $0 = b_1 e^{\lambda.0} + b_2 e^{-\lambda.0}$ and $0 = b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies 0 = b_1 + b_2$ and $b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies b_1 = b_2 = 0$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So again we reject case-II, when $k > 0$.

Case-III When $k < 0$, we can take $k = -\lambda^2 (say)$, then first equations reduces to

$$X'' + \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i. \text{ Therefore its solution will be}$$
$$X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

Using boundary conditions $u(0, t) = 0 = u(a, t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ becomes $0 = c_1 \cos(\lambda .0) + c_2 \sin(\lambda .0)$ and
$0 = c_1 \cos(\lambda a) + c_2 \sin(\lambda a) \implies c_1 = 0$ and
$0 = c_2 \sin(\lambda a) = 0$
Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$\implies \sin \lambda a = 0 \implies \lambda a = n\pi \quad n = 1, 2, 3, ...$$

Thus $\lambda = \dfrac{n\pi}{a}, n = 1, 2, 3, ...$
Hence non-zero solution $X_n(x)$ are given by

$$(c_2)_n \sin\left(\frac{n\pi x}{a}\right) \tag{8}$$

Similarly the solution corresponding to the equation $T'' + \lambda^2 T = 0$ is

$$T_n(t) = (c_3)_n \cos\frac{n\pi ct}{a} + (c_4)_n \sin\frac{n\pi ct}{a} \tag{9}$$

Hence the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right) \tag{10}$$

Where $E_n = ((c_2)_n(c_3))$ and $F_n = ((c_2)_n(c_4)_n)$ are new arbitrary constants.

In order to find a solution which also satisfy $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$, We differentiate equation (10) w.r.t. t,

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{a}\left(\frac{-n\pi c}{a}E_n\sin\frac{n\pi ct}{a}+\frac{n\pi c}{a}Fn\cos\frac{n\pi ct}{a}\right)\right\} \tag{11}$$

Put $t=0$ in equation (10) and (11) and using initial equation $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{a} \tag{12}$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c F_n}{a} \sin \frac{n\pi x}{a} \tag{13}$$

Which are Fourier sin series of expansion $f(x)$ and $g(x)$, respectively. Accordingly we get

$$E_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \tag{14}$$

and

$$F_n = \frac{2}{n\pi c} \int_0^a g(x) \sin \frac{n\pi x}{a} dx \tag{15}$$

Hence the required solution is given by the equation (10) where E_n and F_n are given by the equation (14) and (15).

Thanks !!!