Mathematics-II (Complex Variable)
Lecture Notes
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by

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WORKING RULE: TO CONSTRUCT AN ANALYTIC FUNCTION BY MILNE THOMSON METHOD

Case I. When u is given

Step-1: Find $\frac{\partial u}{\partial x}$ and equate it to $\phi_1(x, y)$.

Step-2: Find $\frac{\partial u}{\partial y}$ and equate it to $\phi_2(x, y)$.

Step-3: Replace x by z and y by 0 in $\phi_1(x, y)$ to get $\phi_1(z, 0)$.

Step-4: Replace x by z and y by 0 in $\phi_2(x, y)$ to get $\phi_2(z, 0)$.

Step-5: Find $f(z)$ by the formula

$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c$

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Example

If $u = x^2 - y^2$, find a corresponding analytic function.

Solution: Here given that $u = x^2 - y^2$. So that $\frac{\partial u}{\partial x} = 2x = \phi_1(x, y)$ and $\frac{\partial u}{\partial y} = -2y = \phi_2(x, y)$. On replacing x by z and y by 0,

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we have

$$\begin{aligned} f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c \\ &= \int (2z) dz + c \\ &= z^2 + c \end{aligned}$$

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Show that $e^x(x\cos y - y\sin y)$ is a harmonic function. Find the analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part.

Solution: Here $v = e^x(x\cos y - y\sin y)$

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \psi_2(x, y)(say), \qquad (1)$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \psi_1(x, y)(say), \qquad (2)$$

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$$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y). \quad (4)$$

Adding equation (3) and (4), we have

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Now putting $x = z$and $y = 0$ in (1) and (2),we get
$\psi_2(z,0) = ze^z + e^z$ and $\psi_1(z,0) = 0$
Hence by Milne-Thomson method, we have

$$\begin{aligned} f(z) &= \int [\psi_1(z,0) + i\psi_2(z,0)]\, dz + c \\ &= \int (0 + i(ze^z + e^z))dz + c \\ &= i(ze^z - e^z + e^z) + c \\ &= ize^z + c \end{aligned}$$

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$\psi_2(z, 0) = ze^z + e^z$ and $\psi_1(z, 0) = 0$
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f(z) &= \int [\psi_1(z,0) + i\psi_2(z,0)]\, dz + c \\
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