Mathematics-II (Complex Variable)
Lecture Notes
May 6, 2020

by

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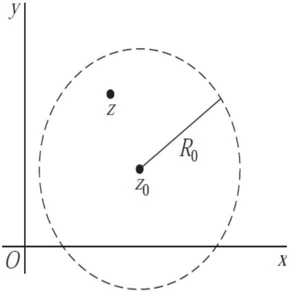
Theorem: Suppose that a function $f(z)$ is analytic throughout a disk $|z-a|<R$, centered at a and with radius R. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad (|z-a|<R)$$

where

$$a_n = \frac{f^n(a)}{n!} \qquad (n = 0, 1, 2, ...).$$


y
z
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i.e. $f(z) =$
$$f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^n(a) + \ldots.$$
This series is called **Taylor's Series** of $f(z)$ about $z = a$.
If $a = 0$, then the series
$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \ldots + \frac{z^n}{n!} f^n(0) + \ldots$ is called **Maclaurin's Series** of $f(z)$ about $z = 0$.

Example

Obtain the Taylor's series expansion of the function $f(z) = \dfrac{1}{z^2 + (1+2i)z + 2i}$ about $z = 0$.

Solution: Here the given function is $f(z) = \dfrac{1}{z^2 + (1+2i)z + 2i}$. This function can be written as

$$f(z) = \frac{1}{(z+2i)(z+1)} = \frac{1}{(2i-1)(z+1)} + \frac{1}{(1-2i)(z+2i)} = \frac{1}{(2i-1)}(z+1)^{-1} + \frac{1}{(1-2i)}(z+2i)^{-1}$$

$$= \frac{1}{(2i-1)}(z+1)^{-1} + \frac{1}{2i(1-2i)}\left(1+\frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{(2i-1)}\left[1-z+z^2-z^3+...\right] + \frac{1}{(2i+4)}\left[1-\frac{z}{2i}-\frac{z^2}{4}-\frac{z^3}{8i}+...\right]$$

After simplifying we get the required expansion.

Theorem: Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z-a| < R_2$, centered at a, and let C denote any positively oriented simple closed contour around a and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

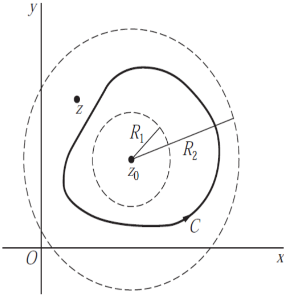
$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$
$$(R_1 < |z-a| < R_2)$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad (n = 0, 1, 2, ...).$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}dz \qquad (n = 1, 2, 3, ...).$$


y
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Definition

Laurent's series: An expansion of the function $f(z)$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

is called Laurent's series expansion. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ is called **Principal Part** of the function $f(z)$ at $z = 0$.

Example

Obtain the Laurent's series expansion of the function $f(z) = \dfrac{1}{(z+1)(z+3)}$, which is valid for

(a) $1 < |z| < 3$ (b) $|z| > 3$

(c) $0 < |z+1| < 2.$

Solution: Here the given function is $f(z) = \dfrac{1}{(z+1)(z+3)}$.

Resolving this function into partial fractions, we get

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right).$$

(a) For $1 < |z| < 3$:

Since $|z| > 1$ and $|z| < 3$, the above fractions can be written as

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2z}\left(\frac{1}{1+1/z}\right) - \frac{1}{2}\frac{1}{3}\left(\frac{1}{1+z/3}\right).$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}.$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \ldots\right].$$

(b) For $|z| > 3$:

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) =$$

$$\frac{1}{2z}\left(\frac{1}{1+1/z}\right) - \frac{1}{2z}\left(\frac{1}{1+3/z}\right).$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}.$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + ...\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + ...\right].$$

$$= \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + ...\right] - \frac{1}{2}\left[\frac{1}{z} - \frac{3}{z^2} + \frac{3^2}{z^3} - \frac{3^3}{z^4} + ...\right].$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \dots.$$

(c) For $|z+1| < 2$:

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+1+2}\right).$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(1 + \frac{z+1}{2}\right)^{-1} =$$

$$\frac{1}{2(z+1)} - \frac{1}{4}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + ...\right]$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - ....$$

Example

Expands $f(z)=\dfrac{z}{(z^2-1)(z^2+4)}$, in $1<|z|<2$.

Solution: Here the given function is $f(z)=\dfrac{z}{(z^2-1)(z^2+4)}$, where $1<|z|<2$ or $1<|z|^2<4$. Resolving this function into partial fractions, we get

$$f(z)=\frac{z}{5}\left(\frac{1}{z^2-1}-\frac{1}{z^2+4}\right)$$

Since $|z|^2 > 1$ and $|z|^2 < 4$, the above fractions can be written as

$$\frac{z}{5z^2}\left(\frac{1}{1-1/z^2}\right)-\frac{z}{5.4}\left(\frac{1}{1+z^2/4}\right)=$$
$$\frac{1}{5z}\left(1-\frac{1}{z^2}\right)^{-1}-\frac{z}{20}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{5z}\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+...\right]-\frac{z}{20}\left[1-\frac{z^2}{4}+\frac{z^4}{4^2}-\frac{z^6}{4^3}+...\right].$$

$$=\frac{1}{5}\left[\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^5}+\frac{1}{z^7}+...\right]-\frac{1}{20}\left[z-\frac{z^3}{4}+\frac{z^5}{4^2}-\frac{z^7}{4^3}+...\right].$$

Which is the Laurent's expansion of $f(z)$in $1 < |z| < 2$.

Example

Expands $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$, in $1 < |z+1| < 3.$

Solution: Let $z+1 = u$, then the given function is $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ can be written as $f(u) = \dfrac{7(u-1)-2}{u(u-1)(u-1-2)} = \dfrac{7u-9}{u(u-1)(u-3)}$, where $1 < |u| < 3$. Resolving this function into partial fractions, we get

$$f(u) = -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

Since $|u| > 1$ and $|u| < 3$, the above fractions can be written as

$$-\frac{3}{u} + \frac{1}{u}\left(\frac{1}{1-1/u}\right) + \frac{1}{3}\left(\frac{2}{u/3-1}\right) = -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\ldots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\frac{u^3}{3^3}+\ldots\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \frac{1}{u^4} + \ldots - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \ldots\right)$$

$$= -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + \ldots - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \ldots\right).$$

Which is the Laurent's expansion of $f(z)$in $1 < |z+1| < 3$.

Thanks !!!