Mathematics-II (Numerical Methods)
Lecture Notes
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by

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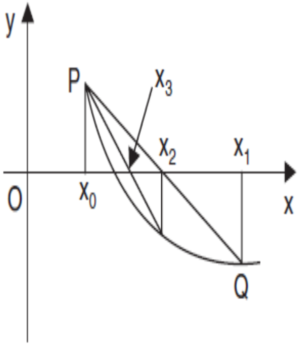
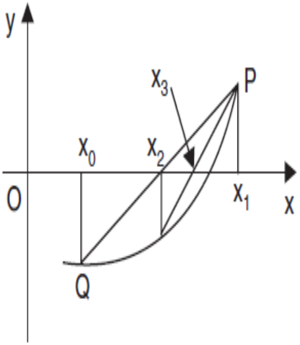
Regula-Falsi method: At the start of all iterations of the method, we require the interval in which the root lies. Let the root of the equation $f(x) = 0$, lie in the interval (x_{k-1}, x_k), that is, $f_{k-1}f_k < 0$, where $f(x_{k-1}) = f_{k-1}$, and $f(x_k) = f_k$.





Then, $P(x_{k-1}, f_{k-1})$, $Q(x_k, f_k)$ are points on the curve $f(x) = 0$. Draw a straight line joining the points P and Q. The line PQ is taken as an approximation of the curve in the interval $[x_{k-1}, x_k]$. The equation of the line PQ is given by

$$\frac{y-f_k}{f_{k-1}-f_k} = \frac{x-x_k}{x_{k-1}-x_k}$$

The point of intersection of this line PQ with the $x-$axis is taken as the next approximation to the root. Setting $y = 0$, and solving for x, we get

$$x = x_k - \left(\frac{x_{k-1}-x_k}{f_{k-1}-f_k}\right) f_k = x_k - \left(\frac{x_k-x_{k-1}}{f_k-f_{k-1}}\right) f_k$$

Thus the $(k+1)^{th}$ iteration will be

$$x_{k+1} = x_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) f_k = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$$

This method is also called **linear interpolation method** or **chord method** or **false position method**.

Example

Locate the intervals which contain the positive real roots of the equation $x^3 - 3x + 1 = 0$. Obtain these roots correct to three decimal places, using the method of false position.

Solution: We form the following table of values for the function $f(x)$.

x	0	1	2	3
$f(x)$	1	-1	3	19

There is one positive real root in the interval $(0, 1)$ and another in the interval $(1, 2)$. There is no real root for $x > 2$ as $f(x) > 0$, for all $x > 2$.

First, we find the root in $(0,1)$. We have $x_0 = 0, x_1 = 1, f_0 = f(x_0) = f(0) = 1, f_1 = f(x_1) = f(1) = -1$.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(0)(-1) - (1)(1)}{(-1) - (1)} = \frac{1}{2} = 0.5$$

Now, $f_2 = f(x_2) = f(0.5) = -0.375$. Since, $f(0)f(0.5) < 0$, the root lies in the interval $(0, 0.5)$.

$$x_3 = \frac{x_0 f_2 - x_2 f_0}{f_2 - f_0} = \frac{(0)(-0.375) - (0.5)(1)}{(-0.375) - (1)} = 0.36364$$

Now, $f_3 = f(x_3) = f(0.36364) = -0.04283$. Since, $f(0)f(0.36364) < 0$, the root lies in the interval $(0, 0.36364)$.

$$x_4 = \frac{x_0 f_3 - x_3 f_0}{f_3 - f_0} = \frac{(0)(-0.04283) - (0.36364)(1)}{(-0.04283) - (1)} = 0.34870$$

Now, $f_4 = f(x_4) = f(0.34870) = -0.00370$. Since, $f(0)f(0.34870) < 0$, the root lies in the interval $(0, 0.34870)$.

$$x_5 = \frac{x_0 f_4 - x_4 f_0}{f_4 - f_0} = \frac{(0)(-0.00370) - (0.34870)(1)}{(-0.00370) - (1)} = 0.34741$$

Now, $f_5 = f(x_5) = f(0.34741) = -0.00030$. Since, $f(0)f(0.34741) < 0$, the root lies in the interval $(0, 0.34741)$.

$$x_6 = \frac{x_0 f_5 - x_5 f_0}{f_5 - f_0} = \frac{(0)(-0.00030) - (0.34741)(1)}{(-0.00030) - (1)} = 0.347306$$

Now, $|x_6 - x_5| = |0.34730 - 0.34741| = 0.0001 < 0.0005$.
The root has been computed correct to three decimal places. The required root can be taken as $x = x_6 = 0.347306$. We may also give the result as 0.347, even though x_6 is more accurate. Note that the left end point $x = 0$ is fixed for all iterations.

Now, we compute the root in $(1,2)$. We have $x_0 = 1, x_1 = 2, f_0 = f(x_0) = f(1) = -1, f_1 = f(x_1) = f(2) = 3$.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(1)(3) - (2)(-1)}{(3) - (-1)} = 1.25$$

Now, $f_2 = f(x_2) = f(1.25) = -0.796875$. Since, $f(1.25)f(2) < 0$, the root lies in the interval $(1.25, 2)$.

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(1.25)(3) - (2)(-0.796875)}{(3) - (-0.796875)} = 1.407407$$

Now, $f_3 = f(x_3) = f(1.407407) = -0.434437$. Since, $f(1.407407)f(2) < 0$, the root lies in the interval $(1.407407, 2)$.
Similarly, we get $x_4 = 1.482367$, $x_5 = 1.513156$, $x_6 = 1.525012$, $x_7 = 1.529462$, $x_8 = 1.531116$, $x_9 = 1.531729$, $x_{10} = 1.531956$.
Now, $|x_{10} - x_9| = |1.531956 - 1.53179| = 0.000227 < 0.0005$.
The root has been computed correct to three decimal places.
The required root can be taken as $x = x_{10} = 1.531956$. Note that the right end point $x = 2$ is fixed for all iterations.

Example

Find the root correct to two decimal places of the equation $xe^x = \cos x$, using the method of false position.

Solution: Define $f(x) = \cos x - xe^x = 0$. We form the following table of values for the function $f(x)$.

x	0	1
$f(x)$	1	-2.17798

A root of the equation lies in the interval $(0, 1)$. Let $x_0 = 0, x_1 = 1$.

Using the method of false position, we obtain the following results.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(0)(-2.17798) - (1)(1)}{(-2.17798) - (1)} = 0.31467$$

Now, $f_2 = f(x_2) = f(0.31467) = 0.51986$. Since, $f(0.31467)f(1) < 0$, the root lies in the interval $(0.31467, 1)$.

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(0.31467)(-2.17798) - (1)(0.51986)}{(-2.17798) - (0.51986)} = 0.44673$$

Now, $f_3 = f(x_3) = f(0.44673) = 0.20354$. Since, $f(0.44673)f(1) < 0$, the root lies in the interval $(0.44673, 1)$.

$$x_4 = \frac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = \frac{(0.44673)(-2.17798) - (1)(0.20354)}{(-2.17798) - (0.20354)} = 0.49402$$

Now, $f_4 = f(x_4) = f(0.49402) = 0.07079$. Since, $f(0.49402)f(1) < 0$, the root lies in the interval $(0.49402, 1)$.

$$x_5 = \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4} = \frac{(0.49402)(-2.17798) - (1)(0.07079)}{(-2.17798) - (0.07079)} = 0.50995$$

Now, $f_5 = f(x_5) = f(0.50995) = 0.02360.$. Since, $f(0.50995)f(1) < 0$, the root lies in the interval $(0.50995, 1)$.

Similarly we get, $x_6 = 0.51520$, $x_7 = 0.51692$.
Now, $|x_7 - x_6| = |0.51692 - 0.51520| = 0.00172 < 0.005$. The root has been computed correct to two decimal places. The required root can be taken as $x = x_7 = 0.51692$. Note that the right end point $x = 2$ is fixed for all iterations.

Thanks !!!