Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Newton-Raphson method:

Let a root of $f(x) = 0$ lie in the interval (a, b). Let x_0 be an initial approximation to the root in this interval. The Newton-Raphson method to find this root is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \text{ provided } f'(x_k) \neq 0$$

This method is called the Newton-Raphson method or simply the **Newton's method**. The method is also called the **tangent method**.

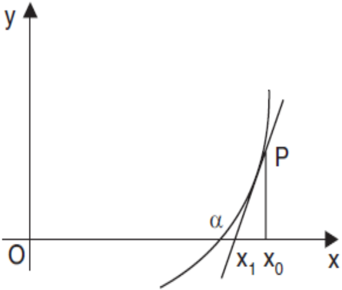

Figure: Newton-Raphson method

Example

Perform four iterations of the Newton's method to find the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.

Solution: We have $f(0) = 1$, $f(1) = -3$. Since, $f(0)f(1) < 0$, the smallest positive root lies in the interval $(0, 1)$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5}, \qquad k = 0, 1, 2, ...$$

Let $x_0 = 0.5$. We have the following results.

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(0.5)^3 - 1}{3(0.5)^2 - 5} = 0.176471,$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(0.176471)^3 - 1}{3(0.176471)^2 - 5} = 0.201568,$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 5} = \frac{2(0.201568)^3 - 1}{3(0.201568)^2 - 5} = 0.201640,$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 5} = \frac{2(0.201640,)^3 - 1}{3(0.201640,)^2 - 5} = 0.201640.$$

Therefore, the root correct to six decimal places is $x \equiv 0.201640$.

Example

Using Newton-Raphson method solve $x\log_{10} x = 12.34$ with $x_0 = 10$.

Solution: Here $f(x) = x\log_{10} x - 12.34$. Then $f'(x) = \log_{10} x + \frac{1}{\log_e 10} = \log_{10} x + 0.434294$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k\log_{10} x_k - 12.34}{\log_{10} x_k + 0.434294} = \frac{(0.434294)x_k + 12.34}{\log_{10} x_k + 0.434294}, \qquad k = 0, 1, 2, ...$$

Let $x_0 = 10$. We have the following results.

$$x_1 = \frac{(0.434294)x_0 + 12.34}{\log_{10} x_0 + 0.434294} = \frac{(0.434294)(10) + 12.34}{\log_{10} 10 + 0.434294} = 11.631465.$$

$$x_2 = \frac{(0.434294)x_1 + 12.34}{\log_{10} x_1 + 0.434294} = \frac{(0.434294)(11.631465) + 12.34}{\log_{10}(11.631465) + 0.434294} = 11.594870.$$

$$x_3 = \frac{(0.434294)x_2 + 12.34}{\log_{10} x_2 + 0.434294} = \frac{(0.434294)(11.594870) + 12.34}{\log_{10}(11.594870) + 0.434294} = 11.594854.$$

We have $|x_3 - x_2| = |11.594854 - 11.594870| = 0.000016$. Therefore, We may take $x \equiv 11.594854$ as the root correct to four decimal places.

Example

Derive the Newton's method for finding $1/N$, where $N > 0$. Hence, find $1/17$, using the initial approximation as $(i) \quad 0.05, \quad (ii) \quad 0.15$. Do the iterations converge

Solution: Let $x = \dfrac{1}{N} \implies N = \dfrac{1}{x}$. Define a function $f(x) = \dfrac{1}{x} - N$ so that $f'(x) = -\dfrac{1}{x^2}$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{\dfrac{1}{x_k} - N}{\left(-\dfrac{1}{x_k^2}\right)} = x_k + \left[x_k - Nx_k^2\right] = 2x_k - Nx_k^2, \qquad k = 0, 1, 2, \ldots$$

(i) With $N = 17$, and $x_0 = 0.05$, we obtain the sequence of approximations

$$x_1 = 2x_0 - Nx_0^2 = 2(0.05) - 17(0.05)^2 = 0.0575.$$

$$x_2 = 2x_1 - Nx_1^2 = 2(0.0575) - 17(0.0575)^2 = 0.058794.$$

$$x_3 = 2x_2 - Nx_2^2 = 2(0.058794) - 17(0.058794)^2 = 0.058823.$$

$$x_4 = 2x_3 - Nx_3^2 = 2(0.058823) - 17(0.058823)^2 = 0.058823.$$

Since, $|x_4 - x_3| = 0$, the iterations converge to the root. The required root is 0.058823.

(ii) With $N = 17$, and $x_0 = 0.15$, we obtain the sequence of approximations

$$x_1 = 2x_0 - Nx_0^2 = 2(0.15) - 17(0.15)^2 = -0.0825.$$

$$x_2 = 2x_1 - Nx_1^2 = 2(-0.0825.) - 17(-0.0825.)^2 = -0.280706.$$

$$x_3 = 2x_2 - Nx_2^2 = 2(-0.280706) - 17(-0.280706)^2 = -1.900942.$$

$$x_4 = 2x_3 - Nx_3^2 = 2(-1.900942) - 17(-1.900942)^2 = -65.23275.$$

We find that $x_k \to -\infty$ as k increases. Therefore, the iterations diverge very fast. This shows the importance of choosing a proper initial approximation.

Example

Derive the Newton's method for finding the q^{th} root of a positive number N, $N^{1/q}$, where $N > 0, q > 0$. Hence, compute $17^{1/3}$ correct to four decimal places, assuming the initial approximation as $x_0 = 2$.

Solution: Let $x = N^{1/q} \implies N = x^q$. Define a function $f(x) = x^q - N$ so that $f'(x) = qx^{q-1}$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^q - N}{qx^{q-1}} = \frac{(q-1)x_k^q + N}{qx^{q-1}}, \qquad k = 0, 1, 2, ...$$

For computing $17^{1/3}$, we have $q = 3$ and $N = 17$. Hence, the method becomes

$$x_{k+1} = \frac{(3-1)x_k^3 + 17}{3x_k^{3-1}} = \frac{2x_k^3 + 17}{3x_k^2}, \qquad k = 0, 1, 2, \ldots$$

With $x_0 = 2$, we obtain the following results.

$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = \frac{2(2)^3 + 17}{3(2)^2} = 2.75,$$

$$x_2 = \frac{2x_1^3 + 17}{3x_1^2} = \frac{2(2.75)^3 + 17}{3(2.75)^2} = 2.582645,$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = \frac{2(2.582645)^3 + 17}{3(2.582645)^2} = 2.571332,$$

$$x_4 = \frac{2x_3^3 + 17}{3x_3^2} = \frac{2(2.571332)^3 + 17}{3(2.571332)^2} = 2.571282.$$

Since, $|x_4 - x_3| = |2.571282 - 2.571332| = 0.00005.$, We may take $x = 2.571282$ as the required root correct to four decimal places.

Thanks !!!